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The quantisation and measurement of momentum observables

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Abstract. Mackey's scheme for the quantisation of classical momenta generating complete vector fields (complete momenta) is introduced, the differential operators corresponding to these momenta are introduced and discussed, and an isomorphism is shown to exist between the subclass of first-order self-adjoint differential operators, whose symmetric restrictions are essentially self-adjoint, and the complete classical momenta. Difficulties in the quantisation of incomplete momenta are discussed, and a critique given. Finally, in an attempt to relate the concept of completeness to measurability, concepts of classical and quantum *global* measurability are introduced, and shown to require completeness. These results afford strong physical insight into the nature of complete momenta, and lead us to suggest a quantisability condition based upon global measurability.

1. Introduction

Mackey (1963) has demonstrated that each classical momentum P associated with a complete vector field X on M generates a one-parameter group of transformations of M, which in turn generates a one-parameter group of unitary transformations U of the set $L^2(M)$ of square-integrable functions on M. The quantisation scheme of Mackey amounts to the identification of the generator of U with the quantised momentum $Q(P)$ (Wan and Viazminsky 1977, 1979). $Q(P)$, being the generator of U, is automatically self-adjoint. Let us call a classical momentum P a complete momentum if the associated vector field X is complete. We then have a quantisation scheme for complete momenta.

This is an important and general result, which nevertheless is far from encompassing the quantisation problem. Such an analysis requires the presence of each of the following elements:

(i) the explicit determination of the existing unique quantum observable corresponding to a complete momentum,

(ii) the discussion of the quantisation of momenta which do not generate complete vector fields,

(iii) an analysis of the dequantisation of quantum momenta, and

(iv) the construction of a theory of measurement of the quantisable momenta, both in the classical and the quantum cases.

This last is necessary if the theoretical construction is to be physically meaningful.

This paper is the outcome of an attempt to construct such a self-contained general scheme of quantisation by seeking to answer as far as possible the foregoing questions, and to explore the inter-relationships which obtain between them. Our analysis leads to the conclusion that the rather abstract and formal quantisation condition on P in Mackey's scheme does have a direct physical foundation, that is, the quantisability condition of P may be traced to its global measurability, a concept to be introduced and discussed in detail in the second half of this paper.

2. The quantisation of momentum observables

2.1. The quantisation of complete momenta

Let the classical configuration space be an N-dimensional Riemannian manifold M with metric g^{ij}. The phase space is then the cotangent bundle T^*M with coordinatisation (x^i, p_i), where p_i are generalised momenta conjugate to coordinates x^i. We shall confine ourselves to classical momentum observables of the form $P=\xi^i(x)\ p_i$ where $\xi^i(x)\in C^\infty(M)$. P is associated with the vector field $X=\xi^i(x)\,\partial/\partial x^i$.

A complete momentum is quantised according to

Theorem 1. (Mackey 1963) On the quantisation of complete momenta: If P is a complete momentum observable, then the symmetric operator $Q_0(P)$ on the Hilbert space $L^2(M)$ defined by

$$Q_0(P)=-\mathrm{i}\hbar(X+\tfrac{1}{2}\operatorname{div}X), \tag{1}$$

with domain

$$DQ_0(P)=\{\psi\in L^2(M)\colon \psi\in C_0^\infty(M),\ Q_0(P)\psi\in L^2(M)\}, \tag{2}$$

where $C_0^\infty(M)$ denotes the class of infinitely differentiable functions of compact support on M, is essentially self-adjoint, and hence possesses a unique self-adjoint extension $Q(P)$ (Varadarajan 1970, Hermann 1978, Abraham and Marsden 1978). $Q(P)$ is postulated to be the quantum analogue of P.

The explicit expression for the operators $Q(P)$ is given by

Theorem 2. (appendix 1) *Explicit representation of $Q(P)$.* The operator $Q(P)$ introduced in theorem 1 is given explicitly by

$$Q(P)=-\mathrm{i}\hbar(D_X+\tfrac{1}{2}\operatorname{div}X), \tag{3}$$

with domain

$$DQ(P)=\{\psi\in L^2(M)\colon \psi\in C^1(X,M),\ Q(P)\psi\in L^2(M)\}, \tag{4}$$

where D_X is the Lie derivative with respect to X (Loomis and Sternberg 1968) and $C^1(X,M)$ is the set of functions on M whose Lie derivative with respect to X exists.

Expressions (3) and (4) may be given the following working interpretations:

(i) If $\psi\in C^1(M)$, then $D_X\psi=X\psi=\xi^i\partial\psi/\partial x^i$. Hence for a $C^1(M)$ function ψ in $DQ(P)$ we have

$$Q(P)\psi=-\mathrm{i}\hbar(X+\tfrac{1}{2}\operatorname{div}X)\psi. \tag{5}$$

However, since $\psi \in C^1(X, M) \not\Rightarrow \psi \in C^1(M)$, it is misleading in our present context to write

$$D_X = X = \xi^i \partial/\partial x^i$$

as the right-hand expression implicitly implies a domain of operation consisting of functions once-differentiable with respect to every coordinate x^i.

(ii) In a coordinate chart x'^i in which $X = \partial/\partial x'^1$ we have $\psi \in DQ(P) \Rightarrow \psi$ is once-differentiable with respect to x'^1, and

$$Q(P)\psi = -\mathrm{i}\hbar(\partial/\partial x'^1 + \tfrac{1}{2}\partial(\ln g^{1/2})/\partial x'^1)\psi, \tag{6}$$

where $g = \det|g_{ij}|$ in coordinates x'^i.

(iii) At critical points where $X = 0$ (Brickell and Clark 1970) we have $D_X = 0$. This means that at critical points the function ψ as an element of $L^2(M)$ need not be differentiable at all.

As an illustration consider the case of the z component angular momentum P_z in $M = \mathcal{R}^3$. Its associated vector field is

$$L_z = x\partial/\partial y - y\partial/\partial x,$$

x, y, z being the usual global Cartesian coordinates in $\mathcal{R}^3$. L_z is complete and hence possesses the quantum analogue

$$Q(P_z) = -\mathrm{i}\hbar D_{L_z},$$

with a domain given by (4). If $\psi \in DQ(P_z)$ and $\psi \in C^1(M)$, then

$$Q(P_z)\psi = -\mathrm{i}\hbar(x\partial/\partial y - y\partial/\partial x)\psi.$$

However, the widely used equality

$$Q(P_z) = -\mathrm{i}\hbar(x\partial/\partial y - y\partial/\partial x)$$

is incorrect, since $DQ(P_z)$ contains functions differentiable with respect to neither x nor y. .The correct differential expression is given locally by

$$Q(P_z) = \begin{matrix} 0 & \text{at the origin} \\ -\mathrm{i}\hbar\,\partial/\partial\phi & \text{elsewhere} \end{matrix}$$

where ϕ, the local azimuthal angle, corresponds to x'^1 in (6). The origin is a critical point at which no restriction of differentiability of any kind is made on any $\psi \in DQ(P_z)$.

The above results provide us with a scheme for quantising complete momenta. This quantisation scheme can be reversed because of the following:

Theorem 3. On the dequantisation problem. The operator $Q_0(P)$ with domain $DQ_0(P)$ as defined in theorem 1 is essentially self-adjoint iff all integral curves of X, except possibly those originating from a set of measure zero in M, are complete (Abraham and Marsden 1978). Let $\{Q(P)\}_c$ denote the set of self-adjoint operators $Q(P) = Q_0^\dagger(P)$, where $Q_0(P)$ satisfy theorem 3, and let $\{P\}_c$ be the set of momenta complete except possibly on a set of measure zero in M. A bijection exists between $\{P\}_c$ and $\{Q(P)\}_c$ given by the operation of quantisation $Q\colon \{P\}_c \to \{Q(P)\}_c$ and by the operation of dequantisation $Q^{-1}\colon \{Q(P)\}_c \to \{P\}_c$. Thus the operations of quantisation and dequantisation give rise to difficulty only for incomplete momenta, that is momenta associated with incomplete vector fields.

2.2. The quantisation of incomplete momenta

The question now arises: what are the quantum analogues of incomplete momenta? For an incomplete momentum P, theorems 1 and 3 imply that $Q_0(P)$ is not essentially self-adjoint. Hence either (1) $Q_0(P)$ has no self-adjoint extensions, which we interpret as meaning that $Q(P)$ does not exist, or else (2) $Q_0(P)$ has many self-adjoint extensions, when the problem reduces to whether any, and if so which, extension is the quantum analogue $Q(P)$ of P. We may shed light on case (2) above by considering an example of a momentum observable associated with a particle moving in the configuration space $M=\{x: x\in(-1,1)\}$ with metric $g=1/(1-x^2)^2$. This configuration space constitutes a complete Riemannian manifold, (Bishop and Crittenden 1964), that is, every geodesic is infinitely extensible in both directions. This is physically important since in such a manifold a classical particle can execute free motion along a geodesic indefinitely. Hence we avoid the difficulty encountered by the usual model of an infinite square well in which even classical motion suffers abrupt discontinuities at the edges of the well. The completeness of the configuration space as a Riemannian manifold is of significance in quantum theory as seen in the following.

Theorem 4. (appendix 2) The symmetric operator $Q_0(H)$ on $L^2(M)$ defined by

$$Q_0(H)=-\frac{\hbar^2}{2m}\nabla^2=-\frac{\hbar^2}{2m}g^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}g^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}$$

with domain

$$DQ_0(H)=\{\psi\in L^2(M): \psi\in C_0^\infty(M),\ Q_0(H)\psi\in L^2(M)\},$$

is essentially self-adjoint if and only if the space $M=\{x: x\in(a,b)\}$ with metric g is a complete Riemannian manifold.

The unique self-adjoint extension $Q(H)$ of $Q_0(H)$ is the free quantum Hamiltonian corresponding to the free classical Hamiltonian H. The choice of the Laplacian operator to obtain $Q(H)$ can be justified by the consideration of Wan and Viazminsky (1977) in the case of a space of constant curvature of which our present M is an example. The explicit expression for $Q(H)$ is

$$Q(H)=-\frac{\hbar^2}{2m}g^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}g^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}$$

with domain

$$DQ(H)=\{\psi\in L^2(M): \psi\in C^2(M),\ Q(H)\psi\in L^2(M)\}.$$

We see that the completeness of M leads to the uniqueness of the quantum Hamiltonian $Q(H)$. We have therefore a well-defined model with no ambiguity in $Q(H)$. There is no need to employ boundary conditions to define the quantum Hamiltonian as is the case in the usual model of an infinite square well.

Returning to our model with $M=\{x: x\in(-1,1)\}$, $g=1/(1-x^2)^2$, we consider the momentum p_x associated with $X=\partial/\partial x$, that is p_x is the momentum conjugate to x. It is readily verified that p_x is incomplete and that $Q_0(p_x)$ has uncountably many self-adjoint extensions $Q_\beta(p_x)$ given by

$$Q_\beta(p_x)=-\mathrm{i}\hbar[\mathrm{d}/\mathrm{d}x+x/(1-x^2)]$$

with domain

$$DQ_\beta(p_x) = \{\phi \in L^2(M): \phi \in C^1(M),\ Q_\beta(p_x)\phi \in L^2(M),$$
$$\lim_{x\to -1} (1-x^2)^{-1/2}\phi(x) = \exp(\mathrm{i}\beta) \lim_{x\to 1} (1-x^2)^{-1/2}\phi(x)\},$$

where $\beta \in [0, 2\pi)$ is a fixed constant. The spectrum of $Q_\beta(p_x)$ is discrete and non-degenerate with normalised eigenfunctions

$$\phi_{\beta,n} = (\tfrac{1}{2}(1-x^2))^{1/2} \exp(-\mathrm{i}k_n x), \qquad n = 0, \pm 1, \pm 2, \ldots,$$

with eigenvalues

$$\hbar k_n = \tfrac{1}{2}\hbar\beta + n\pi\hbar.$$

Since $\phi_{\beta,n} \notin DQ_\gamma(p_x)$, $\beta \neq \gamma$, it follows that $\langle \phi_{\beta,n} | Q_\gamma(p_x) | \phi_{\beta,n} \rangle$ is undefined; physically this results in an infinite average value of $Q_\gamma(p_x)$ (Wan and Viazminsky 1977), and the assumption that $Q_\beta(p_x)$ and $Q_\gamma(p_x)$ are both observables corresponding to attributes of the physical system gives rise to an inconsistency. Hence at most one of the $Q_\beta(p_x)$ can be a system observable. It is difficult to justify the claim that a particular $Q_\beta(p_x)$ is the true quantum analogue of p_x, since there would seem to be no *a priori* criterion to decide among the various $\beta \in [0, 2\pi)$. It is, moreover, impossible to determine any particular choice of β by recourse to experiment, since the spectrum of $Q_\beta(P)$ may be arbitrarily well approximated by that of $Q_\gamma(P)$, $\gamma \simeq \beta$. Hence the experimental spectrum cannot single out $Q_\beta(P)$ *a posteriori*.

We have demonstrated some of the difficulties arising from an attempt to quantise incomplete momenta. A fresh look at this problem from a fundamentally different viewpoint will be presented in the remainder of this paper. In passing we should mention that extensive discussions on the operator i d/dx and its self-adjoint extensions on $L^2(a, b)$ are given by Akhiezer and Glazman (1966) and by Reed and Simon (1972, 1975).

3. The measurement of momenta

3.1. *Classical momentum measurement in Euclidean space*

Consider initially the simplest case of the measurement of the linear momentum p of a free particle moving in a one-dimensional Euclidean space $M_1 = \mathcal{R}$, p being the momentum conjugate to a global Cartesian coordinate y. A measurement of p necessarily involves a non-zero, though perhaps very small, displacement of the measured particle, as is sufficiently apparent from the definition of p, namely $p = m \lim_{\Delta t \to 0} (y(t+\Delta t) - y(t))/\Delta t$, where $y(t)$ is the particle trajectory as a function of the time t. In what follows we shall adopt the impulsive measurement model of Aharanov and Safko (1975). We elect to measure the momentum of the above test particle by measuring the displacement of a particle of momentum p' conjugate to its coordinate y' with which it interacts in accordance with the Hamiltonian

$$H = p^2/2m + p'^2/2m' + \omega(t)pp', \tag{7}$$

where $\omega(t)$ is given by

$$\omega(t) = \begin{cases} \omega_0, & t \in (0, T), \\ 0, & t \notin (0, T). \end{cases}$$

The equations of motion of the test and measuring particles are therefore

$$y(t)-y(0)=pt/m+\omega tp',$$
$$y'(t)-y'(0)=p't/m'+\omega tp.$$

Taking the impulsive limit $\omega_0 T\to\gamma>0$ as $T\to 0$, we deduce that

$$\Delta y=y(T)-y(0)=\gamma p', \tag{8}$$
$$\Delta y'=y'(T)-y'(0)=\gamma p. \tag{9}$$

Thus we see that, if the value of γ characteristic of the interaction is known, then p is measurable in terms of the displacement (recoil) $\Delta y'$ of a reference particle. We note particularly that only the recoil $\Delta y'$ is required so that the position of the particle under test need not be known, provided only that the recoil is measurable. This analysis shows that (in this particular example) the momentum p may be measured in a manner independent of and without the knowledge of position.

Suppose now the particle is confined to the configuration space $M_2=(a, b)$. We can employ the same measurement model (7) in M_2 to obtain (8) and (9) provided $y(t)$, $y'(t)$ lie in (a, b) for $t\in(0, T)$. This proviso drastically changes the fundamental nature of the measuring process. A given measuring device characterised by pre-fixed values of γ and p' may no longer be used to measure p without a knowledge of y, that is if y is too near the boundary a, the value Δy may be so large as to violate the condition $y\in(a, b)$. Furthermore, for any fixed value of γ a value of p exists such that $\Delta y'$ exceeds the value $b-a$. To sum up, we see that generally a p measurement causes uncertainties in y and y'. Depending on the geometric property of the manifold M the uncertainties may be subject to certain position-dependent constraints, rendering a p measurement generally y dependent as well as y' dependent.

A similar situation obtains in $\mathscr{R}^n$ if Cartesian coordinates y^i, y'^i and their conjugate momenta p_i, p'_i are considered. The interaction Hamiltonian is

$$H=(1/2m)\eta^{ij}p_ip_j+(1/2m')\eta^{ij}p'_ip'_j+\omega(t)\eta^{ij}p_ip'_j, \tag{10}$$

where $\eta^{ij}=\delta^{ij}$ is the Euclidean metric. The recoil displacements of the interacting particles are related to their momentum values by

$$\Delta y^i=\gamma p'_i, \qquad \Delta y'^i=\gamma p_i. \tag{11}$$

3.2. *Classical momentum measurement in Riemannian space*

The problem herein lies in the measurement of a general function $P=\xi^i(\bar{x})\bar{p}_i$ on T^*M, $(\bar{x}^i, \bar{p}_i)$ being a coordinatisation on T^*M. In the neighbourhood of any point in M where $\xi^i\neq 0$ coordinates x^i exists such that $P=p_1$, the momentum conjugate to x^1. Hence P is just a simple momentum variable which can be ascertained without having to measure canonical variables $\bar{x}^i$, $\bar{p}_i$ simultaneously. From now on it is sufficient to consider the measurement of a momentum p_1 conjugate to a coordinate x^1. For simplicity we shall assume x^1 to be a global coordinate.

The model measuring process is a kind of collision between two free particles, one the test particle described by undashed quantities, the other the reference (measuring) particle described by dashed quantities; the measured parameter is the reference particle recoil. As the total Hamiltonian describing the collision, we elect the natural extension of (7) and (10), namely

$$H = (1/2m)g^{ij}(x)p_ip_j + (1/2m')g^{ij}(x')p'_ip'_j + \omega(t)g^{ij'}(x, x')p_ip'_j, \tag{12}$$

where $g^{ij'}(x, x')$ is the parallel propagator (Synge 1971). The parallel propagator introduced by Synge may not be unique. However, this will not affect our following analysis which involves only local properties of $g^{ij'}$. The solution of Hamilton's equations of motion generated by (12) for the global trajectories of the particles would, in general, prove a formidable task. Fortunately it is sufficient for our purposes to consider the local motion in the neighbourhood of the collision in terms of the Cartesian coordinates y_0^i about some neighbouring fixed point m_0. The equations of transformation connecting the coordinate systems are (Synge and Schild 1966)

$$y_0^i = b_j^i(x^j - a^j),$$

where b_j^i is a matrix of constants and the point m_0 has coordinates $x^j = a^j$. It is, moreover, possible to elect b_j^i such that (appendix 3)

$$y_0^1 = b_1^1(x^1 - a^1). \tag{13}$$

In terms of these coordinates the Hamiltonian (12) becomes

$$H = (1/2m)\eta^{ij}p_i^0p_j^0 + (1/2m')\eta^{ij}p_i'^0p_j'^0 + \omega(t)\eta^{ij}p_i^0p_j'^0, \tag{14}$$

where p_i^0 and $p_i'^0$ are momenta conjugate to y_0^i and $y_0'^i$ respectively. The equations of transformation (13) yield simply $p_1 = b_1^1p_1^0$. Hamilton's equations of motion yield, by (14), $\Delta y_0^1 = \gamma p_1^0$, whence, provided the reference particle is so aligned that $p_i'^0 = 0$, $i \neq 1$, we may deduce the equation connecting p_1 and the measured displacement $\Delta y'^1$ as

$$p_1 = b_1^1\Delta y_0'^1/\gamma. \tag{15}$$

All other coordinate displacements Δy^j, $j \neq 1$, are zero.

We have now established a local measurement procedure for p_1. Our aim is to set up a position-independent measuring process. We shall introduce a notion of global measurability to achieve this goal.

3.3. *Global measurability, completeness and quantisability*

In order not to obscure various arguments by mathematical technicalities, let us consider the generalised momentum P conjugate to an arbitrary global coordinate x in the one-dimensional Euclidean manifold $M = \mathcal{R}$. In terms of a Cartesian coordinate y we have $P = \xi(y)p$, where p is the momentum conjugate to y and $\xi = dy/dx \neq 0$ (to be definite we assume $\xi(y) > 0 \,\forall\, y \in \mathcal{R}$). We now apply the model measuring process discussed in the preceding section to measure P. The simplicity of our present case enables us to see clearly the nature of the approximation involved. Our model measuring process as embodied in (13), (14), (15) amounts to (i) a straightforward measurement of p and (ii) the approximation of $P = \xi(y)p$ by $P = \xi(y_0)p$. The approximation may be made arbitrarily accurate since we can choose γ and p' to produce very small recoil Δy. In other words, we do have a perfectly valid measuring process for P.

There is, however, a serious problem caused by the local nature of the measuring process. In our model the measuring device is the reference particle (locally characterised by its momentum p' conjugate to its locally Cartesian coordinate y') which would interact with the measured particle with a certain fixed value of γ. Our measuring device is a local system and only its local properties are relevant. Let us call two measuring devices sited in the neighbourhoods of two different points m_1, m_2 in M identical if their respective local properties at m_1, m_2 are the same, i.e. they possess the same values of p' and γ. Now recall that our aim is to set up a basically position-independent measuring process for P. But the local nature of our measuring process and measuring device seems to go against that. This turns out to be only an apparent paradox as seen in the following detailed analysis. At the outset let us agree that by a position-independent measuring process for P we mean the possibility of measuring P with a pre-elected accuracy using identical measuring devices independent of where the particle is in M. We can now investigate this possibility. Firstly, identical measuring devices will produce a common recoil Δy independent of where the particle is, that is, independent of y. So the problem reduces to the investigation of error caused by replacing $\xi(y)$ by $\xi(y_0)$ where $y \in (y_0, y_0+\Delta y)$. This error may be characterised by the standard deviation $\Delta_s\xi_0$ of ξ as a function of y over the interval $(y_0, y_0+\Delta y)$ defined by (see for example Peslak 1979)

$$(\Delta_s\xi_0)^2 = \frac{1}{\Delta y}\int_{\Delta y} (\xi - \bar{\xi}_0)^2 \,\mathrm{d}y, \qquad \bar{\xi}_0 = \frac{1}{\Delta y}\int_{\Delta y} \xi \,\mathrm{d}y.$$

If $\Delta_s\xi_0$ is bounded for every $y_0 \in \mathscr{R}$ (Δy being constant), then we have a position-independent measuring process for P, so that a common measuring device can measure P to within a common tolerance, supremum $((\Delta_s\xi)p)$, for any fixed Cartesian momentum p, independent of the position of the particle. Obviously the possibility of achieving this depends on the nature of P, that is the property of $\xi(y)$. Let us call a momentum variable P globally measurable if for P the above position-independent measuring process applies. The significance of global measurability of P is seen in theorem 5, whose somewhat intricate proof is given in appendix 4.

Theorem 5. on completeness and global measurability of p.

(i) Let $y_n = n\Delta y$, where $\Delta y > 0$ is a fixed number and $n = \pm 1, \pm 2, \ldots$, be a sequence of points on $\mathscr{R}$. Let $X = \xi(y)\,\mathrm{d}/\mathrm{d}y$, $\xi(y) > 0$, be an incomplete vector field on $\mathscr{R}$. Then

$$\overline{\lim}\, \Delta_s\xi_n = \infty,$$

where $\overline{\lim}$ denotes the limit superior as $n \to +\infty$ or $-\infty$.

(ii) A momentum $P = \xi p$, $\xi \neq 0$, is globally measurable only if it is complete.

An incomplete momentum is hence not globally measurable since the uncertainty $\Delta_s\xi_n$ is unbounded. This is then the intrinsic measurement-theoretic significance of a complete momentum. Since completeness of P is the quantisability condition of P we have succeeded in tracing this abstract mathematical condition to its physical origin, that is, the quantisability of P is intrinsically related to the global measurability of P. A momentum P is quantisable if it is globally measurable, that is if P is measurable in a position-independent manner. This result is pleasing especially in view of the fact that the concept of global measurability may be carried over into quantum theory, a result which we shall demonstrate in the following section.

3.4. Quantum global measurability

The basic impulsive measurement model applies in quantum theory according to Aharanov and Safko (1975). With a parallel analysis one may expect to obtain similar results. However we shall adopt a more general approach based on the uncertainty principle. We shall consider a momentum $P=\xi(y)p$, $\xi(y)>0$ in $\mathcal{R}$, y being a global Cartesian coordinate. Firstly we introduce the concept of quantum global measurability. Let $y_n = n\Delta y$, Δy being fixed. Let ϵ be a fixed positive number much less than Δy. Let $(\epsilon_{n^-}, \epsilon_{n^+})$ be any interval in (y_{n-1}, y_{n+1}) with $\epsilon_{n^+}-\epsilon_{n^-}=\epsilon$. Furthermore let Φ_n be a set of localised wavefunctions given by $\Phi_n=\{\phi\in L^2(R)\cap DQ(P)$: $(\epsilon_{n^-},\epsilon_{n^+})\subset \operatorname{supp}(\phi)\subset(y_{n-1},y_{n+1})$, $\langle\phi|\phi\rangle=1\}$, and let $\Delta_s Q(P)_{\phi_n}=[\langle\phi_n|Q(P)^2|\phi_n\rangle-\langle\phi_n|Q(P)|\phi_n\rangle^2]^{1/2}$, $\phi_n\in\Phi_n$. A quantised observable $Q(P)$ is said to be globally measurable if the sequence of uncertainties $\Delta_s Q(P)_{\phi_n}$ (with respect to a chosen sequence of wavefunctions $\phi_n\in\Phi_n$) is bounded. We shall now establish a link between completeness and the global measurability of $Q(P)$. From the general uncertainty principle we have

$$\Delta_s Q(P)\Delta_s y \geqslant \tfrac{1}{2}\hbar|\langle\xi\rangle|$$

for a quantum momentum of the form

$$Q(P)=-\mathrm{i}\hbar(X+\tfrac{1}{2}\operatorname{div}X),$$

where $X=\xi(y)\,\mathrm{d}/\mathrm{d}y$ and $\xi(y)>0$. For a localised wavefunction we have $\Delta_s y\leqslant\Delta y$. It is not too difficult then to establish

Theorem 6. (appendix 5) *On completeness and global measurability of $Q(P)$. $Q(P)$, $P=\xi p$, $\xi\neq 0$,* is globally measurable only if P is complete. The close link between completeness of P and the global measurability of P and $Q(P)$ is now seen. Here we also unearth a fundamental cause of trouble associated with the quantisation of incomplete momenta, that is, in addition to the ambiguity in the choice of its quantum counterpart we have also that any quantum counterpart chosen is not globally measurable.

It is worth pointing out here that the concept of global measurability is in keeping with the notion that space is locally the same everywhere and is indistinguishable by local measurements.

3.5. Generalisation to higher dimensional manifolds

Our analysis in the previous two sections has been performed in a one-dimensional manifold in order to explain the ideas involved without excessive mathematical complication. The analysis can be readily extended to higher dimensional manifolds in view of the following:

Theorem 7. (appendix 6) Let X be an incomplete vector field without a critical point in a complete Riemannian manifold M. There exists an integral curve $\sigma_0(t)$ of X from some point $m_0=\sigma_0(0)\in M$ such that

(i) $\sigma_0(t)$ is neither closed nor has end points,

(ii) $\sigma_0(t)$ is isometric to $\mathcal{R}$,

(iii) there is an open covering of $\sigma_0(t)$ where coordinates x^i exist in terms of which we have $x^1\in(-\infty,\infty)$, $g_{ij}=\delta_{ij}$ and $X=\xi(x^1)\partial/\partial x^1$.

Here x^1 is really the signed metric distance along σ_0 with $x^1=0$ at m_0, $x^1(\sigma_0(t))>0$ for $t>0$ and $x^1(\sigma_0(t))<0$ for $t<0$. Hence x^1 can serve as a local Cartesian coordinate with origin at m_0. We see that the problem reduces to a one-dimensional problem along the coordinate x^1. Consequently results identical to those embodied in theorem 5 part (ii) and theorem 6 are obtained with reasonable facility, global measurability being interpreted as the existence of a supremum of ΔP_n (i.e. $(\Delta_s \xi_n)p$ or $\Delta_s Q(P)_n$) for each integral curve σ_0 of the vector field. It should be noted that we do not require the existence of an upper bound (for fixed Δy, say) of the suprema of ΔP_n for all integral curves. For this much stronger statement, while still admitting the above theorems, needlessly excludes many complete momenta from the measurable class.

4. Conclusion

The analyses of the foregoing sections have identified two severe difficulties associated with the retention of incomplete momenta as quantisable objects:

(i) The absence of any uniquely and explicitly known general procedure for their quantisation, and the difficulty in distinguishing various choices of $Q(P)$ by measurement of their eigenvalue spectra.

(ii) Their global immeasurability, especially quantum mechanically. These two factors respectively reflect upon complementary aspects of the process of subjecting the theory to experimental test. It should be pointed out that the objections raised against incomplete momenta are based on their global properties. We do not claim that they have no local meaning; indeed it may be that they are very useful in discussing local physical effects. It remains true, however, that the behaviour of a quantum momentum observable is decisively influenced by the global character of the space so that a fully meaningful observable must be well-defined in a global sense.

In view of what has been said we are tempted to put forward the following physical quantisability axiom: *A* momentum P is quantisable iff P is globally measurable (classically or quantum mechanically). Obviously this assertion is closely related to Mackey's quantisability condition. However, with the existing formulation of global measurability, this new quantisability axiom is more restrictive than that based upon completeness. But we do nevertheless feel that further work along this line on the relationship between quantisability and measurability with an aim of establishing a physical quantisability axiom based on a concept of global measurability would be highly fruitful, and we hope to be able to report new results in the near future.

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Appendix 1. Theorem 2 on the explicit representation of $Q(P)$

Assume initially that the vector field X has no critical points, and denote $Q_0(P)$ and $Q_0^\dagger(P)$ by Q_0 and $Q_0^\dagger$ respectively. Introduce the auxiliary operator $Q_1(P) = Q_1$ by

$$Q_1 = -i\hbar(D_X + \tfrac{1}{2}\operatorname{div} X), \tag{A1.1}$$

$$DQ_1 = \{\psi \in L^2(M) | \psi \in C_0^1(X, M),\ Q_1\psi \in L^2(M)\}. \tag{A1.2}$$

Then since Q_0 is essentially self adjoint, and Q_1 is symmetric, $Q_0^\dagger = Q_1^\dagger = Q$, so that it suffices to find the adjoint of Q_1.

For each $\psi \in DQ_1^\dagger$ let ζ and η be two scalar functions (possibly complex) on M satisfying

$$\operatorname{div} \zeta X = Q_1^\dagger \psi, \qquad \operatorname{div} \eta X = \tfrac{1}{2} i\hbar\psi \operatorname{div} X. \tag{A1.3}$$

Let $\phi \in DQ_1$ be a function of compact support in M; the divergence theorem then gives

$$\langle \phi | \operatorname{div} \zeta X\rangle = -\langle D_X\phi|\zeta\rangle, \qquad \langle \phi | \operatorname{div} \eta X\rangle = -\langle D_X\phi|\eta\rangle. \tag{A1.4}$$

Moreover $\langle \phi | Q_1^\dagger \psi\rangle = \langle Q_1\phi|\psi\rangle = +i\hbar\langle X\phi|\psi\rangle + \langle\phi|\operatorname{div}\zeta X\rangle$, and hence, combining (A1.3) and (A1.4) we have

$$\langle D_X\phi|(+i\hbar\psi + \zeta - \eta)\rangle = 0.$$

Now in a coordinate system in which $X = \partial/\partial x^1$, proposition A1.1 given at the end of this appendix implies that

$$i\hbar\psi = \eta - \zeta + r(x^2, \ldots, x^n)/g^{1/2}. \tag{A1.5}$$

Moreover equations (A1.3) may be solved explicitly for η and ζ, giving, upon substitution into (A1.5)

$$i\hbar\psi = \frac{+i\hbar}{2g^{1/2}}\int^{x^1} \psi(\operatorname{div} X) g^{1/2}\, dx^1 - \frac{1}{g^{1/2}}\int^{x^1} (Q_1^\dagger\psi) g^{1/2}\, dx^1 + f(x^2, \ldots, x^n)/g^{1/2},$$

whence ψ is absolutely continuous with respect to x^1 and

$$Q_1^\dagger\psi = Q_0^\dagger\psi = -i\hbar\left(\frac{\partial\psi}{\partial x^1} + \frac{1}{2}\frac{\partial \ln g^{1/2}}{\partial x^1}\right). \tag{A1.6}$$

$$\Rightarrow Q_0^\dagger = Q_1^\dagger = -i\hbar(D_X + \tfrac{1}{2}\operatorname{div} X),$$
$$DQ_0^\dagger = DQ_1^\dagger = \{\psi \in L^2(M) : \psi \in C^1(X, M),\, Q_0^\dagger\psi \in L^2(M)\}. \tag{A1.7}$$

Finally suppose that X possesses critical points, and let N denote the set of these; then, if N is of non-zero measure, we may show explicitly (by considering $\phi \in C_0^1(X, N) \subset DQ_1$) that $Q_1^\dagger\psi = 0$ at every point in N, so that equations (A1.3) are consistent. Moreover (A1.3) cannot restrict the values of η or ζ at the points of N, so that (A1.5) is no longer valid in N, but it nevertheless remains so at the points of $M - N$, so that the final result (A1.7) remains true.

Proposition A1.1.

$$\langle\psi|D_X\phi\rangle = 0 \quad \forall\phi \in C_0^\infty(M) \Rightarrow \psi \in C^1(X, M) \text{ and } D_X(\psi g^{1/2}) = 0.$$

Proof. Let U_α be an open covering of M with the property that in each open set U_α where $X \neq 0$, a local chart x^i exists such that $X = \partial/\partial x^1$ in U_α. This is possible as U_α can be chosen sufficiently small. Let $C_0^\infty(U_\alpha)$ be a restriction of $C_0^\infty(M)$ to U_α, i.e. $f \in C_0^\infty(U_\alpha) \Leftrightarrow f \in C_0^\infty(M)$ and $\operatorname{supp} f \subset U_\alpha$. Consider the equation $\langle\psi|D_X f\rangle = 0$, $f \in C_0^\infty(U_\alpha)$. Explicitly we have

$$\langle\psi|D_X f\rangle = \int_M \bar\psi \frac{\partial f}{\partial x^1} g^{1/2}\, dx = \int_{U_\alpha} \bar\psi\frac{\partial f}{\partial x^1} g^{1/2}\, dx = 0. \tag{A1.8}$$

These equations may be regarded as equations in the Hilbert space $L^2(U_\alpha)$. In other words we can define an operator $D_X^{(\alpha)} = \partial/\partial x^1$ in $L^2(U_\alpha)$ with domain $C_0^\infty(U_\alpha)$. Let $\psi^{(\alpha)}$ be the restriction of ψ to U_α, i.e. $\psi^{(\alpha)} = \psi(m)$ for every $m \in U_\alpha$ and $\psi^{(\alpha)} = 0$ for every

$m \notin U_\alpha$. Then (A1.8) implies that $\psi^{(\alpha)}$, regarded as an element of $L^2(U_\alpha)$, belongs to the domain of the adjoint of $D_X^{(\alpha)}$. A simple extension of the result of Wan and Viazminsky shows that $\psi^{(\alpha)} \in C^1(X, U_\alpha)$. Then $C_0^\infty(U_\alpha)$ being dense in $L^2(U_\alpha)$ implies $\partial(\psi^{(\alpha)} g^{1/2})/\partial x^1 = D_X^{(\alpha)}(\psi^{(\alpha)} g^{1/2}) = 0$ in U_α. If X has no critical points then this conclusion may be extended to the entire M to obtain the final result.

The result remains valid even if X has critical points since $D_X = 0$ at critical points.

Appendix 2. Proof of theorem 4

Effect a coordinate transformation from x to y in general by $y = \int_{x_0}^x g^{1/2}\,\mathrm{d}x$, $x_0, x \in (a, b)$ and in particular for $M = (-1, 1)$, $g = (1 - x^2)^{-2}$ by $y = -\frac{1}{2}\ln[(1-x)/(1+x)]$. Then $y \in (-\infty, \infty)$ and in terms of y the metric is 1. This shows that the configuration space is simply the Euclidean space $\mathscr{R}$ and that

$$
\begin{aligned}
&Q_0(H) = -(\hbar^2/2m)\,\mathrm{d}^2/\mathrm{d}y^2 = Q_0(p_y)Q_0(p_y),\\
&Q_0^\dagger(H) \supseteq Q_0^\dagger(p_y)Q_0^\dagger(p_y) = Q(p_y)Q(p_y),\\
&Q_0^{\dagger\dagger}(H) \subseteq (Q(p_y)Q(p_y))^\dagger = Q(p_y)Q(p_y)\\
\Rightarrow\ &Q_0^\dagger(H) = Q_0^{\dagger\dagger}(H), \text{ hence result.}
\end{aligned}
$$

A related theorem applicable to a general Riemannian manifold is available in Abraham and Marsden (1978).

Appendix 3

Given arbitrary coordinates x^i we can introduce a normal coordinate system $\bar{x}^i$ (Synge and Schild 1966) such that

$$\bar{x}^1 = x^1, \qquad g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \bar{g}_{11}\,\mathrm{d}\bar{x}^1\,\mathrm{d}\bar{x}^1 + \bar{g}_{rs}\,\mathrm{d}\bar{x}^r, \mathrm{d}\bar{x}^s, \qquad r, s \neq 1.$$

Furthermore

$$\bar{p}_1 = (\partial x^i/\partial \bar{x}^1)p_i = p_1.$$

For local Cartesian coordinates y^i we can choose to have

$$y_0^1 = b_1^1(\bar{x}^1 - a^1) = b_1^1(x^1 - a).$$

Then

$$\bar{p}_1 = (\partial y^i/\partial \bar{x}^1)p_i^0 = b_1^1 p_1^0, \qquad b_1^1 = (|\bar{g}_{11}|)^{1/2} = (|g_{11}|)^{1/2}.$$

Appendix 4. Theorem 5 on completeness and global measurability of classical P

We present a sequence of propositions leading to theorem 5, and employ the following notation:

Let $F(y) > 0$ be a continuous function on $\mathscr{R}$, and let $\{y_n\}$ be the sequence described in theorem 5. Let $\underline{\lim}$, $\overline{\lim}$, lim denote the limit inferior, limit superior, and usual limit respectively.

We employ the following known results:

(i) Every limit point of a sequence $\{u_n\}$ may be regarded as a limit point of a suitably chosen subsequence $\{u_{n_i}\}$ of $\{u_n\}$ (Knopp 1928). The convergence of $\Sigma_n u_n$ implies that

(ii) $\lim u_n = 0$ (Widder 1961),

(iii) if $u_n > 0$ and $u_{n+1} \leq u_n$, then $\lim nu_n = 0$ (Bromwich 1908), and

(iv) if $u_n > 0$ and $\lim u_n/u_{n+1} = 1$, then $\overline{\lim}\, n(u_n/u_{n+1} - 1) \geq 1$ (Bromwich 1908).

Proposition A4.1. Let f_n, the local mean over $[y_n, y_{n+1}]$ of $F(y)$, be defined by

$$\int_{y_n}^{y_{n+1}} F(y)\,\mathrm{d}y = f_n(y_{n+1} - y_n) = f_n\,\Delta y;$$

then

$$\int_0^\infty F(y)\,\mathrm{d}y < \infty \Rightarrow \underline{\lim}\, nf_n = 0.$$

Proof. Noting (ii), construct a subsequence of $\{f_n\}$ for which (iii) applies.

Proposition A5.2. Let f'_n, the local reciprocal mean, be defined by

$$\frac{1}{f'_n} = \frac{1}{\Delta y}\int_{y_n}^{y_{n+1}} \frac{\mathrm{d}y}{F(y)};$$

then

$$\int_0^\infty F(y)\,\mathrm{d}y < \infty \Rightarrow \underline{\lim}\, nf'_n = 0.$$

Proof. Employ the Schwarz inequality (Widder 1961) to demonstrate that $f'_n \leq f_n$, when this proposition is immediately corollary to A4.1.

Proposition A4.3. Let $\delta_n = \Delta y/(f_{n+1}^{-1} - f_n^{-1})$, where f_n is as above; then $\int_0^\infty F(y)\,\mathrm{d}y < \infty$ implies that $\{\delta_n\}$ has 0 as a limit point.

This proposition remains true if f_n is replaced by f'_n in the statement.

Proof.

$$\delta_n = \frac{\Delta y\, nf_n}{n(f_n/f_{n+1} - 1)},$$

and $\lim nf_n = 0$.

We may distinguish three cases:

(i) $\lim f_n/f_{n+1} = A \neq 1$, when clearly $\lim \delta_n = 0$,

(ii) $\lim f_n/f_{n+1} = 1$, when we consider a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ as selected in the proof of proposition A4.1 and apply (iv),

(iii) f_n/f_{n+1} oscillates indefinitely, when we may construct suitable subsequences to demonstrate the result.

We now introduce the concept of local standard deviation $\Delta_s\xi$ of a function $\xi(y)$ on $\mathscr{R}$ over the interval $[a, b]$ defined by (see, say, Peslak 1979),

$$(\Delta_s\xi)^2 = \frac{1}{b-a}\int_a^b (\xi - \bar{\xi})^2\,\mathrm{d}y, \qquad \bar{\xi} = \frac{1}{b-a}\int_a^b \xi\,\mathrm{d}y.$$

We then have the following propositions:

Proposition A4.4.

$$(\Delta_s \xi)^2 \geqslant \tfrac{1}{4}(\bar{\xi}_+ - \bar{\xi}_-)^2,$$

where

$$\bar{\xi}_- = \frac{2}{b-a}\int_a^c \xi \, \mathrm{d}y, \qquad \bar{\xi}_+ = \frac{2}{b-a}\int_c^b \xi \, \mathrm{d}y,$$

and $c = \frac{1}{2}(b+a)$.

Proof. Introduce

$$\xi'(y) = \begin{cases} \bar{\xi}_-, & y \in [a, c] \\ \bar{\xi}_+, & y \in (c, b] \end{cases};$$

then by direct calculation $(\Delta_s \xi')^2 = \frac{1}{4}(\bar{\xi}_+ - \bar{\xi}_-)^2$, and from the Schwarz inequality $(\Delta_s \xi)^2 \geqslant (\Delta_s \xi')^2 \geqslant 0$.

Theorem A4.1. Let $\xi(y) > 0$ be a continuous function on $\mathscr{R}$ and let

$$(\Delta_s \xi_n)^2 = \frac{1}{2\Delta y}\int_{y_{n-1}}^{y_{n+1}} (\xi - \bar{\xi}_n)^2 \, \mathrm{d}y, \qquad \text{where } \bar{\xi}_n = \int_{y_{n-1}}^{y_{n+1}} \xi(y) \, \mathrm{d}y / 2\,\Delta y.$$

Then

$$\int_0^\infty \frac{\mathrm{d}y}{\xi(y)} < \infty \Rightarrow \overline{\lim}\, \Delta_s \xi_n = \infty.$$

Proof. Let

$$\bar{\xi}_{n^+} = \frac{1}{\Delta y}\int_{y_n}^{y_{n+1}} \xi(y)\,\mathrm{d}y, \qquad \bar{\xi}_{n^-} = \frac{1}{\Delta y}\int_{y_{n-1}}^{y_n} \xi(y)\,\mathrm{d}y, \qquad F(y) = \frac{1}{\xi(y)}$$

$$\frac{1}{f'_n} = \frac{1}{\Delta y}\int_{y_n}^{y_{n+1}} \frac{1}{F}\,\mathrm{d}y = \bar{\xi}_{n^+}, \qquad \frac{1}{f'_{n-1}} = \frac{1}{\Delta y}\int_{y_{n-1}}^{y_n} \frac{1}{F}\,\mathrm{d}y = \bar{\xi}_{n^-}.$$

The sequence $\delta'_{n-1} = \Delta y / [(1/f'_n) - (1/f'_{n-1})] = \Delta y / (\bar{\xi}_{n^+} - \bar{\xi}_{n^-})$ has zero as a limit point if $\int_0^\infty F(y)\,\mathrm{d}y < \infty$ by proposition A4.3, that is the sequence $(\bar{\xi}_{n^+} - \bar{\xi}_{n^-})$ has ∞ as a limit point. The proposition A4.4 implies $\overline{\lim}\, \Delta_s \xi_n = \infty$.

Theorem A4.2.
Let $X = \xi(y)\,\mathrm{d}/\mathrm{d}y$, $\xi(y) > 0$ be an incomplete vector field on $\mathscr{R}$. Then

$$\overline{\lim}\, \Delta_s \xi_n = \infty$$

where $\Delta_s \xi_n$ is as defined in theorem A4.1.

Proof. A direct consequence of theorem A4.1 and proposition A4.3.

Appendix 5. Theorem 6 on completeness and global measurability of $Q(P)$

Let $\Delta_s Q(P)_{\phi_n}$ be the uncertainty with respect to a normalised wavefunction $\phi_n \in \Phi_n$ localised in the interval (y_{n-1}, y_{n+1}).

$$\Delta_s Q(P)_{\phi_n} \geq \hbar |\langle\xi\rangle_n|/4\,\Delta y, \qquad \text{where } \langle\xi\rangle_n = \int_{y_{n-1}}^{y_{n+1}} \phi_n^* \phi_n \xi \, dy.$$

$Q(P)$ being globally measurable implies that the sequence $\int_{y_{n-1}}^{y_{n+1}} \xi \, dy$ must be bounded, since $\int_{y_{n-1}}^{y_{n+1}} \xi \, dy$ is the supremum of $\langle\xi\rangle_n$ over all such localised wavefunctions ϕ_n in (y_{n-1}, y_{n+1}). This implies that $\{\xi_n\}$, $\{\xi_{n^+}\}$, $\{\xi_{n^-}\}$ (introduced in the proof of theorem A4.1) are bounded and that P is complete (see proposition A4.3 and the proof of theorem A4.1).

Appendix 6. Theorem 7

Proposition A6.1. There exists a $\sigma_0(t)$ as described in theorem 6 which is neither closed nor has end points.

Proof. If every maximal integral curve of X is closed X must be complete. For an incomplete X, a $\sigma_0(t)$ exists which is not closed. Such a $\sigma_0(t)$ cannot have end points either, otherwise it would contradict the result that at every point on the curve, $\sigma_0(t)$ may be regarded locally as a coordinate curve (Brickell and Clark 1970).

Proposition A7.2. $\sigma_0(t)$ contains its limit points.

Proof. Use the result that $\sigma_0(t)$ is locally a coordinate curve.

Proposition A7.3. Let $S(t)$ denote the metric distance along the curve $\sigma_0(t)$ described in theorem 7 between the point $m_0 = \sigma_0(0)$ and $m = \sigma_0(t)$, $t > 0$, and let (a, b) be the domain of $\sigma_0(t)$. Then $S(t) \to \infty$ as $t \to b$.

Proof. Let $d(m_1, m_2)$ be the metric distance between the points $m_1 = \sigma_0(t_1)$ and $m_2 = \sigma_0(t_2)$ in the sense of Choquet-Bruhat *et al* (1977). Let $t_n > 0$ be a sequence converging to b. Now, suppose the corresponding sequence $S(t_n)$ converges to a value $S(b) < \infty$. Then $S(t_n)$ is necessarily a Cauchy sequence, i.e. $|S(t_n) - S(t_{n'})| \to 0$ as n, $n' \to \infty$. Consequently the sequence $m_n = \sigma_0(t_n)$ is Cauchy since $|S(t_n) - S(t_{n'})| \geq d(m_n, m_{n'}) \geq 0$. Since our manifold M is assumed to be proper and complete the Hopf–Rinow theorem (Choquet-Bruhat *et al* 1977) operates. The sequence m_n converges to a point $m_b \in M$ as $t_n \to b$. Proposition A6.2 means that m_b must be a point on the curve $\sigma_0(t)$. On the other hand m_b must also be an end point of $\sigma_0(t)$. These contradict proposition A6.1. Hence the premise $S(b) < \infty$ is false.

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